

PP-waves on Superbrane Backgrounds

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In this paper we discuss a method of generating solutions of the Einstein equations on supersymmetric and non-supersymmetric backgrounds. The method involves the embedding of a supersymmetric spacetime into another, curved spacetime. We present three examples with constituent spacetimes which support “charges”, one of which was known previously and the other two of which are new. All of the examples have PP-waves as one of the embedding constituents.

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Supersymmetry is a unifying principle which helps overcome the divergence problem and the “naturalness” problem of the Standard Model [1]. All consistent string theories are supersymmetric [2]. Therefore the study of supersymmetric spacetimes is a worthwhile endeavor because of the intrinsic interest of such spacetimes and because of their importance to other theoretical investigations. There are relatively few spacetimes which possess at least some supersymmetries and even fewer which are maximally supersymmetric, i.e. possess the maximum number of supersymmetries allowed by the dimension of the spacetime [3].

In this letter we present a general method for generating solutions of the Einstein equations on supersymmetric backgrounds. The method also works for non-supersymmetric spaces and raises the possibility of designer solutions of Einstein’s equations. The application of the method studied in the present work involves the embedding of a fractionally supersymmetric spacetime into another spacetime plus a deformation term in the component of the metric for the coordinate which is common to the superbranes of the constituent spacetimes. We discuss three examples of this method, one previously known and two which are new. The supersymmetric examples discussed here are for $D = 11$ supergravity, but the method works for supergravity in any number of dimensions. Although the constituents in the supersymmetric examples are plane parallel gravitational waves propagating on supersymmetric $D = 11$ curved spacetime backgrounds, the method should work for other backgrounds such as type IIA supergravity-superstrings. Throughout the paper capital Latin indices run from 0 to 10, Greek indices from 0 to 2, $(m, n) = 3 \dots 10$, and $(i, j) = 2 \dots 10$. The metric signature is $(-1, 1 \dots 1)$.

Gravitational waves with parallel wave fronts and parallel rays in 11 dimensions have a metric of the form [4]

$$ds^2 = 2 dx^+ dx^- + H(x^i, x^-) (dx^-)^2 + \delta_{ij} dx^i dx^j, \quad (1)$$

where $x^\pm = x_1 \pm x_0$ are light-cone coordinates, and $H(x^i, x^-)$ is a function which is determined from the relation

$$\square H = -\frac{1}{12} |\Phi|^2 \quad (2)$$

in which \square is the Laplacian operator in the transverse Euclidean space \mathbb{E}^9 , and Φ is a 3-form in \mathbb{E}^9 . Φ is related to the 4-form arising from the “charge” on the superbrane by $F_4 = dx^- \wedge \Phi$. The

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Lagrangian density for this system is given by [5]

$$\kappa^2 \mathcal{L} = \frac{1}{2} \sqrt{-g} \left[R - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right] + \frac{1}{2(12)^4} \varepsilon^{UVWMNOPQRST} F_{UVWM} F_{NOPQ} A_{RST}, \quad (3)$$

where $\varepsilon^{UVWMNOPQRST}$ is the eleven-dimensional Levi-Civita tensor density, and A_{MNP} is an anti-symmetric potential with corresponding field strength given by

$$F_{MNPQ} = 4\partial_{[M} A_{NPQ]}. \quad (4)$$

The components of the 4-form F_4 satisfy the field equation

$$\partial_M (\sqrt{-g} F^{MUVW}) + \frac{1}{1152} \varepsilon^{UVWMNOPQRST} F_{MNOP} F_{QRST} = 0 \quad (5)$$

and the metric tensor elements satisfy the Einstein equations

$$R_{MN} - \frac{1}{2} g_{MN} R = \kappa^2 T_{MN} \quad (6)$$

with

$$\kappa^2 T_{MN} = \frac{1}{12} \left(F_M{}^{PQR} F_{NPQR} - \frac{1}{8} g_{MN} F_{PQRS} F^{PQRS} \right). \quad (7)$$

The Killing spinors for this spacetime satisfy the relation

$$(D_M - \Omega_M) \epsilon = 0, \quad (8)$$

in which D_M is the covariant derivative

$$D_M = \partial_M + \frac{1}{4} \omega_M{}^{AB} \Gamma_{AB}, \quad (9)$$

and

$$\Omega_M = \frac{1}{288} F_{PQRS} \left(\Gamma^{PQRS}{}_M + 8 \Gamma^{PQR} \delta_M^S \right). \quad (10)$$

The eleven-dimensional Dirac matrices satisfy the relation $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$ and for convenience we define $\Gamma_{AB..C} = \Gamma_{[A} \Gamma_{B..} \Gamma_{C]}$.

The metric described in Eq. (1) preserves half of the maximum allowed 32 supersymmetries. For certain choices of $H(x^i, x^-)$ the space described by Eq. (1) is maximally supersymmetric. The latter occurs when $H(x^i, x^-)$ is of the form

$$H(x^i, x^-) = \begin{cases} -\frac{1}{9} \mu^2 \delta_{ij} x^i x^j & i, j = 2, 3, 4 \\ -\frac{1}{36} \mu^2 \delta_{ij} x^i x^j & i, j = 5, \dots, 10 \end{cases} \quad (11)$$

$$\Phi = \mu dx^2 \wedge dx^3 \wedge dx^4,$$

where μ is a constant. The non-zero components of the 4-form for this case are $F_{-[234]} = \mu$, where the square brackets indicate complete antisymmetry in the indices. Spacetimes with a null homogeneous flux which describe gravitational waves with a constant F_4 are called homogeneous, plane parallel (HPP) waves. The space described by Eq. (11) is a special case of the Cahen - Wallach spaces [7], which are indecomposable Lorentzian symmetric spaces.

Embedding a PP-wave in an AdS spacetime results in a supersymmetric space which typically preserves $\frac{1}{4}$ of the maximum supersymmetry allowed by the AdS spacetime. For certain choices of $H(x^i, x^-)$ gravitational waves on AdS backgrounds result in so-called “supernumerary” supersymmetries [6], which preserve $\frac{1}{2}$ the maximal number of supersymmetries. For purely gravitational PP-waves on an AdS background the metric is given by

$$ds^2 = dx_2^2 + e^{2ax_2} [2 dx^+ dx^- + H(x^-, x_2, x_m) (dx^-)^2 + \delta_{mn} dx^m dx^n] , \quad (12)$$

where a is a constant. Supersymmetric solutions to the Einstein equations with d -form field strength sources for PP-waves propagating on an AdS spacetime of any dimension D have been discussed in Ref. [6]. For the case of maximal supersymmetry the Killing spinor solution has the form

$$\epsilon = e^{ax_2/2} \left(1 - \frac{ix_m U(x^+) \Gamma_m \Gamma_-}{2} \right) \left(1 + \frac{ie^{-ax_2} U(x^+) \Gamma_-}{2a} \right) \left[1 - \frac{1}{2} \left(1 - e^{-\int U dx^+} \right) \Gamma_+ \Gamma_- \right] \epsilon_0 , \quad (13)$$

where the constant spinor ϵ_0 satisfies the relation $(\Gamma_{x_2} + 1) \epsilon_0 = 0$, and $U(x^+)$ is a function which can be determined from the integrability conditions. Since there is only one condition on ϵ_0 , this spacetime preserves $\frac{1}{2}$ the supersymmetry.

To illustrate our method of generating solutions of the Einstein equations which preserve at least some of the supersymmetries, we consider now the embedding of the PP-wave geometry into the superbrane solution of $D = 11$ supergravity. In Ref. [5] a solution of the coupled supergravity - superbrane equations which preserves $\frac{1}{2}$ of the supersymmetries was obtained. The metric for this space is given by

$$ds^2 = \left(1 + \frac{q}{r^6} \right)^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{q}{r^6} \right)^{1/3} \delta_{mn} dy^m dy^n . \quad (14)$$

The 4-form for the superbrane solution can be written in terms of a 3-form gauge field as given in Eq. (4) with an antisymmetric potential given by

$$A_{\mu\nu\rho} = \pm \frac{1}{3g} \varepsilon_{\mu\nu\rho} \left(1 + \frac{q}{r^6} \right)^{-1} , \quad (15)$$

where q is a constant, $\varepsilon_{\mu\nu\rho}$ is the three-dimensional covariant Levi-Civita symbol, ${}^3g = \det(g_{\mu\nu})$, and $r^2 = \sum_m y_m^2$. The only non-zero 3-form components are

$$A_{[012]} = \pm \left(1 + \frac{q}{r^6} \right)^{-1} , \quad (16)$$

for which

$$F_{[012]m} = \pm \frac{6q x_m}{r^8 (1 + q/r^6)^2} . \quad (17)$$

The Einstein tensor elements are

$$G_{\mu\nu} = (-1)^{\delta_{\mu 0}} \frac{9q^2 r^4}{(r^6 + q)^3} \delta_{\mu\nu} , \quad G_{mn} = \begin{cases} \frac{18q^2 x_m x_n}{r^4 (r^6 + q)^2} & m \neq n \\ -\frac{9q^2 (r^2 - 2x_m^2)}{r^4 (r^6 + q)^2} & m = n . \end{cases} \quad (18)$$

These expressions were obtained for a spacetime which is invariant under $P_3 \times SO(8)$, where P_3 is the $D = 3$ Poincaré group. The Killing spinors can be obtained from a covariant equation similar to Eq. (8). The Dirac matrices for this case are

$$\Gamma_A = (\gamma_\mu \otimes \Gamma_9, \mathbf{I} \otimes \Sigma_m) , \quad (19)$$

where γ_μ and Σ_m are the $D = 3$ and $D = 8$ Dirac matrices, respectively, and $\Gamma_9 = \Sigma_3 \Sigma_4 \dots \Sigma_{10}$. The equations for the Killing spinors result in two nontrivial solutions

$$(1 \pm \Gamma_9) \epsilon(r) = 0, \quad (20)$$

where

$$\epsilon(r) = (1 + q/r^6)^{1/6} \epsilon_0. \quad (21)$$

Each of these solutions preserves half of the maximum possible supersymmetries. To embed the PP-waves in the superbrane background (14), the line element is written as

$$ds^2 = \left(1 + \frac{q}{r^6}\right)^{-2/3} \left\{ 2 dx^+ dx^- - \frac{\mu^2}{2} [x_2^2 - H(r)] (dx^-)^2 + dx_2^2 \right\} + \left(1 + \frac{q}{r^6}\right)^{1/3} \delta_{mn} dy^m dy^n. \quad (22)$$

The function $H(r)$ is determined from Einstein's equations:

$$H(r) = C_1 + \frac{C_2}{r^6} - \frac{q}{4r^4}, \quad (23)$$

where C_1 and C_2 are constants of integration. The 4-form components are assumed to have the same forms as those for the two constituent spaces

$$F_{[-+2]m} = \pm \frac{6q x_m}{r^8 (1 + q/r^6)^2}, \quad F_{-[345]} = \mu. \quad (24)$$

The non-zero Einstein tensor elements are

$$G_{--} = -\frac{\mu^2}{8(r^6 + q)^3} \left[4r^6(r^6 + q)^2 + 9q^3 + 36q^2 x_2^2 r^4 - 36C_1 q^2 r^4 - \frac{36C_2 q^2}{r^2} \right],$$

$$G_{+-} = \frac{9q^2 r^4}{(r^6 + q)^3}, \quad G_{22} = \frac{9q^2 r^4}{(r^6 + q)^2}, \quad G_{mn} = \begin{cases} \frac{18q^2 x_m x_n}{r^4(r^6 + q)^2} & m \neq n \\ -\frac{9q^2(r^2 - 2x_m^2)}{r^4(r^6 + q)^2} & m = n. \end{cases} \quad (25)$$

Although the solution to the Einstein equations found in Eq. (22) does not preserve any supersymmetries for $D = 11$ supergravity, it is possible that other embeddings preserving a fraction of the maximal number of supersymmetries of the two constituent spaces may be found. In the case of PP-waves embedded in an AdS background a solution was found which preserves all 32 of the supersymmetries [4]. However in that case the background has no “charge” supported by a superbrane. Whether or not our method always reduces the number of preserved supersymmetries is a subject for future investigation.

The function $H(r)$ (Eq. (23)), which describes the deformation of the metric tensor element, g_{--} , for the light cone coordinate which is common to both of the charge supporting membranes for the constituent spacetimes, arises from the embedding procedure and is required for consistency between the Einstein tensor and the stress-energy tensor. The deformation term is proportional to the product of the charges, μ and q , of the intersecting membranes. It distorts the wave hypersurfaces defined by the condition $x^+ = \text{constant}$ and the geodesics.

Our method works for nonsupersymmetric backgrounds as well as supersymmetric ones. As an example consider the metric for the p -brane in D dimensions [8] described by ($D = q + p + 2$)

$$ds_p^2 = R^{\Delta/(p+1)} \left[-dx_0^2 + \sum_{i=1}^p (dx^i)^2 \right] + R^{(2-q-\Delta)/(q-1)} dr^2 + r^2 R^{(1-\Delta)/(q-1)} d\Omega_q^2, \quad (26)$$

where

$$\Delta = \sqrt{\frac{q(p+1)}{q+p}}, \quad \text{and} \quad R = \left[1 - \left(\frac{r_0}{r}\right)^{q-1}\right]. \quad (27)$$

Embedding the PP-wave geometry into this background the metric becomes

$$ds^2 = C(r) \left[2 dx^+ dx^- - \mu^2 \bar{x}^2 (dx^-)^2 + \sum_{i=2}^p (dx^i)^2 \right] + E(r) dr^2 + r^2 B(r) d\Omega_q^2, \quad (28)$$

where $\bar{x}^2 = \sum_{i=2}^p (x^i)^2$ and

$$C(r) = R^{\Delta/(p+1)}, \quad B(r) = R^{(1-\Delta)/(q-1)}, \quad E(r) = R^{(2-q-\Delta)/(q-1)}. \quad (29)$$

The non-zero 4-form components for this solution are

$$F_{[-234]} = [2(p-1)\mu^2 C(r)^3]^{1/2} \quad (30)$$

and the non-zero Einstein tensor element is

$$G_{--} = -(p-1)\mu^2. \quad (31)$$

The method for generating solutions to Einstein's equations described in this paper opens up several lines of investigation. We intend to investigate the general procedure for embedding any SUSY geometry into another SUSY geometry. The PP-wave geometry is relatively simple and is easily embedded into a background geometry. We will investigate the applicability of the procedure to more complex spacetimes. For example, we will investigate the possibility that the PP-wave superbrane geometry obtained above can be embedded in an AdS geometry. If the latter is successful, an interesting question is: can the resulting geometry be used to elucidate the AdS/CFT correspondence? Another line of investigation is to study what determines how many supersymmetries are preserved in geometries which are embedded into one another. The superbrane geometry in Eq. (14) can be reduced to a type IIA superstring under the simultaneous reduction of spacetime and worldvolume. The spacetime described by Eq. (22) will be studied to see if a similar reduction can reduce the PP-wave superbrane geometry to a ten-dimensional superstring. In Ref. [9] the perturbative string spectrum was obtained from the gauge theory point of view on a PP-wave. The PP-waves incorporate the first correction to the flat space results for certain states. We can now investigate the question: What spectrum is reproduced from $\mathcal{N} = 4$ Y-M on a PP-wave superbrane geometry?

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